# Enhancing Indoor UAV Localization Accuracy Using Linear Least Squares and Singular Value Decomposition with Tikhonov Regularization

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***Abstract****—Autonomous navigation for Unmanned Aerial Vehicles (UAVs) in GPS-denied indoor environments relies heavily on accurate localization systems. Ultra-Wideband (UWB) technology offers a viable solution through time-of-flight ranging; however, measurement noise and suboptimal anchor geometries often lead to significant localization errors. This paper proposes a robust localization algorithm based on trilateration, converting the non-linear spherical intersection problem into a linear system solved via Linear Least Squares (LLS). To address the numerical instability caused by ill-conditioned matrices—often resulting from coplanar anchor configurations or measurement noise—we implement Singular Value Decomposition (SVD) enhanced with Tikhonov Regularization. Simulation results demonstrate that the proposed method effectively suppresses noise and prevents solution divergence, yielding a stable trajectory reconstruction with a low Root Mean Square Error (RMSE) even under significant Gaussian noise conditions.*

***Keywords****—UAV Localization, Ultra-Wideband (UWB), Singular Value Decomposition, Tikhonov Regularization, Linear Algebra.*

# 1 Introduction

The deployment of Unmanned Aerial Vehicles (UAVs) for indoor applications, such as warehouse management and structural inspection, has grown exponentially. Unlike outdoor environments where Global Positioning System (GPS) signals are reliable, indoor environments require alternative localization strategies to ensure stable autonomous flight [1]. While visual Simultaneous Localization and Mapping (vSLAM) provides high accuracy, it demands significant computational power and is sensitive to lighting conditions. Consequently, Ultra-Wideband (UWB) technology has emerged as a preferred alternative due to its low latency, high update rate, and ability to penetrate obstacles [2].

UWB systems operate by measuring the distance between a mobile tag on the UAV and fixed anchors in the environment. Mathematically, determining the UAV's position $(x, y, z)$ from these distance measurements requires solving a system of quadratic equations, a process known as trilateration. In ideal conditions, the intersection of three spheres yields a unique solution. However, in real-world scenarios, sensor readings are corrupted by Gaussian noise and non-line-of-sight (NLOS) errors, rendering the system overdetermined and inconsistent [3].

To solve this, the problem is typically linearized into the form $Ax = b$ and solved using Least Squares estimation. A critical challenge arises when the geometric configuration of the anchors is suboptimal—for instance, when anchors are placed at similar altitudes (coplanar). In such cases, the matrix $A$ becomes ill-conditioned or singular, causing standard inversion techniques to fail or produce spatially divergent results [4].

This paper addresses these mathematical challenges by proposing a robust estimation algorithm. We utilize Singular Value Decomposition (SVD) to compute the pseudo-inverse of the system matrix, providing a minimum-norm solution for the overdetermined system. Furthermore, we integrate Tikhonov Regularization to dampen the effects of small singular values, thereby stabilizing the solution against measurement noise and geometric singularities [5]. The efficacy of this algebraic approach is validated through Python-based simulations, demonstrating superior stability compared to standard algebraic inversion methods.

# 2 Related Work

The field of indoor UAV localization has evolved significantly, transitioning from basic radio-signal strenght methods to advanced time-of-flight geometric solutions. This section reviews the development of these methods and the algebraic principles underlying them.

## 2.1 Evolution of Indoor Localization Technologies

Early indoor localization systems primarily utilized Received Signal Strength Indicator (RSSI) fingerprinting from WiFi or Bluetooth Low Energy (BLE) networks [1]. While cost-effective, these systems suffer from signal fluctuation due to multipath fading and shadowing, resulting in localization errors often exceeding 2 meters. To address high-precision requirements, Ultra-Wideband

(UWB) technology was introduced. Dardari et al. [2] demonstrated that UWB's large bandwidth provides high time resolution, allowing for centimeter-level accuracy using Time-of-Flight (ToF) ranging, which is critical for the high-speed maneuvers required in drone racing.

### 2.2 Algebraic Foundations of Localization

Mathematically, determining a 3D position from multiple distance measurements is a geometric problem that translates into a System of Linear Equations (SPL). Sir Rinaldi [6] defines an SPL as a collection of linear equations involving the same set of variables, which can be represented compactly as an Augmented Matrix $[A|b]$.

The fundamental methods for solving such systems are rooted in classical linear algebra. As detailed in the lecture materials by sir Rinaldi [6], elementary row operations (OBE) are used to simplify these matrices. Furthermore, sir Rinaldi [7] explains that for a system to have a unique solution, the matrix must be non-singular. The Gauss-Jordan elimination method is traditionally employed to transform the augmented matrix into a reduced row echelon form, allowing for the direct computation of the solution vector $\mathbf{x}$ or the matrix inverse $A^{-1}$ [7].

### 2.3 Limitations of Classical Methods in Real-World Scenarios

While Gaussian and Gauss-Jordan elimination are powerful for consistent systems (where an exact solution exists), they face significant challenges in practical sensor networks. In a real-world drone flight, sensor measurements are corrupted by Gaussian noise, making the system overdetermined and inconsistent ($Ax \approx b$).

Strang [3] highlights that in such overdetermined cases, the simple matrix inverse does not exist. The standard approach is to employ the Normal Equation $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$ to find a Least Squares solution. However, this method is highly sensitive to the condition number of matrix $A$. Golub and Van Loan [4] argue that if the anchor geometry is suboptimal (e.g., coplanar anchors on a ceiling), the matrix $A^TA$ becomes nearly singular (ill-conditioned). In this state, standard inversion techniques amplify even minor sensor noise into massive positional errors.

### 2.4 Regularization Approaches

To mitigate the instability of ill-conditioned matrices, advanced decomposition techniques are required. Singular Value Decomposition (SVD) offers a robust alternative by decomposing the matrix into orthogonal components, isolating the singular values that cause instability [4]. Furthermore, Tikhonov and Arsenin [5] introduced the concept of regularization, which adds a damping factor to the inversion process. This study builds upon these concepts by integrating the algebraic foundations of SPL [6, 7] with SVD and Tikhonov regularization to create a localization algorithm capable of handling the extreme noise and geometric constraints found in the arena.

## 3 Theoretical Background

This section outlines the fundamental principles of linear algebra used in formulating the localization problem, starting from the basic System of Linear Equations (SPL) to the advanced Singular Value Decomposition (SVD) method required for robust estimation.

### 3.1 Fundamentals of Linear Systems

A System of Linear Equations (SPL) is defined as a collection of linear equations involving the same set of variables. In the context of 3D localization, these variables correspond to the spatial coordinates of the drone. A general system of $m$ equations and $n$ variables can be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

This system is linear because the highest power of any variable is 1. It can be compactly represented in matrix notation as $Ax = b$, where $A$ is the $m \times n$ coefficient matrix, $x$ is the variable vector, and $b$ is the constant vector.

An essential tool for analyzing and solving such systems is the **Augmented Matrix**, denoted as $[A|b]$. This matrix combines the coefficient matrix $A$ and the constant vector $b$ into a single entity:

$$[A|b] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix} \tag{2}$$

### 3.2 Standard Solution Methods

Classical linear algebra provides rigorous methods for finding the solution vector $x$, primarily through **Elementary Row Operations (OBE)**. There are three valid operations that do not change the solution set of the system:

1. Multiply a row by a non-zero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

#### 3.2.1 Gaussian and Gauss-Jordan Elimination

The solution to an SPL is obtained by applying OBE to the augmented matrix until it reaches a specific form. **Gaussian Elimination** transforms the matrix into *Row Echelon Form*, followed by backward substitution to solve for variables.

**Gauss-Jordan Elimination** extends this process to convert the matrix into *Reduced Row Echelon Form*, where the leading entry of each nonzero row is 1 and is the only nonzero entry in its column. This allows the solution to be read directly without backward substitution.

While Gaussian elimination provides exact solutions for square matrices, it is ill-suited for overdetermined systems typically found in UWB localization. Therefore, in this study, we employ Singular Value Decomposition (SVD) to obtain the least-squares solution, which effectively mitigates the impact of measurement noise [4].

#### 3.2.2 Matrix Inversion Method

For a square system ($n \times n$) where matrix $A$ is non-singular (has an inverse), the unique solution can be computed using the inverse matrix $A^{-1}$:

$$x = A^{-1}b \tag{3}$$

The inverse $A^{-1}$ itself can be computed using Gauss-Jordan elimination applied to the augmented matrix $[A|I]$ to yield $[I|A^{-1}]$.

However, in practical localization scenarios, the system is often **overdetermined** ($m > n$) due to the presence of multiple anchors ($N > 4$) and is inconsistent due to sensor noise. In such cases, a direct analytical solution via Gaussian elimination or simple inversion does not exist, necessitating approximation methods like Linear Least Squares.

### 3.3 UWB Ranging and Trilateration

Ultra-Wideband (UWB) localization relies on the Time of Flight (ToF) principle to estimate the distance between a tag and anchors. Geometrically, the locus of points at a distance $d_i$ from an anchor $A_i$ forms a sphere. For $N$ anchors, the position is the intersection of $N$ spheres.

Due to measurement noise ($\epsilon$), the spheres do not intersect at a single point. The problem becomes finding a point $\mathbf{p}$ that minimizes the squared error:

$$\min_{\mathbf{p}} \sum_{i=1}^{N} \left(\|\mathbf{p} - \mathbf{a}_i\| - d_i\right)^2 \tag{4}$$

This non-linear least squares problem is computationally expensive. Therefore, algebraic linearization is preferred for real-time UAV applications.

### 3.4 The Linear Least Squares (LLS) Problem

Linearization transforms the geometric problem into an algebraic system $A\mathbf{x} = \mathbf{b}$. The standard analytical solution is given by the normal equations:

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} \tag{5}$$

This method, often called the Moore-Penrose pseudo-inverse, assumes that $(A^T A)$ is invertible.

### 3.5 Ill-Conditioned Matrices and Singularity

In Linear Algebra, a matrix is considered ill-conditioned if its Condition Number, $\kappa(A)$, is very large. The condition number is defined as the ratio of the largest to smallest singular value:

$$\kappa(A) = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{6}$$

If anchors are placed in a coplanar configuration (e.g., all at the same ceiling height), the column vectors of matrix $A$ become linearly dependent. This causes $\sigma_{\min} \to 0$ and $\kappa(A) \to \infty$. In this scenario, the standard inverse (5) amplifies measurement noise, resulting in catastrophic positioning errors.

## 4 Proposed Method

In this study, we formulated the localization problem as a linear system derived from spherical trilateration equations. We applied Singular Value Decomposition (SVD) with Tikhonov Regularization to solve the system robustly against measurement noise.

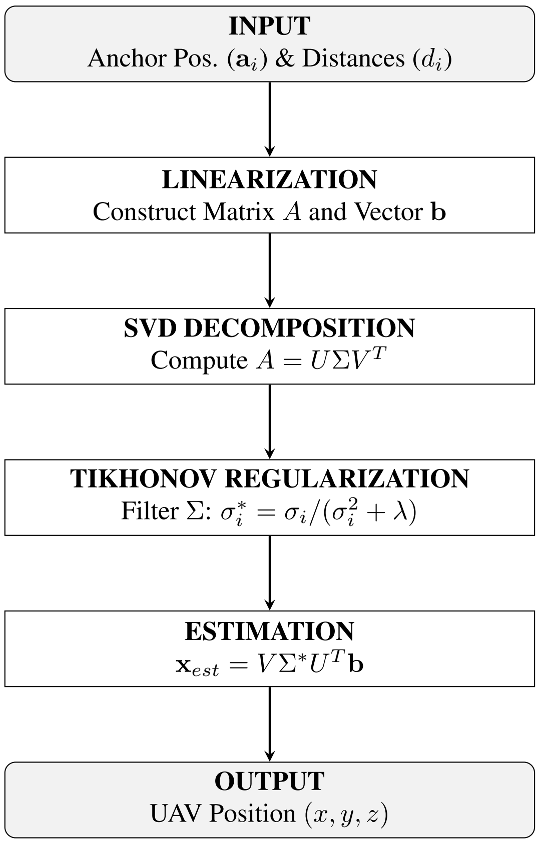


Figure 1: Flowchart of the proposed SVD-based localization algorithm with Tikhonov regularization.

### 4.1 Problem Formulation

Let the unknown position of the UAV be $\mathbf{p} = [x, y, z]^T$. The distance $d_i$ measured by the $i$-th UWB anchor located at $\mathbf{a}_i = [x_i, y_i, z_i]^T$ is given by:

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = d_i^2 \tag{7}$$

Expanding (7) yields the non-linear term $x^2 + y^2 + z^2$. To linearize this, we adopt the pseudo-linear approach where the squared norm is treated as an auxiliary variable. The equation is reformulated as:

$$x_i x + y_i y + z_i z + k = b_i \tag{8}$$

where $k = -0.5(x^2+y^2+z^2)$ and the measurement vector $b_i$ is defined as:

$$b_i = \frac{1}{2}(x_i^2 + y_i^2 + z_i^2 - d_i^2) \tag{9}$$

By stacking equations for $N$ anchors, we obtain the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = [x, y, z, 1]^T$ (treating the 4th component as a slack variable for the constant). The matrix $A$ is constructed as:

$$A = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_N & y_N & z_N & 1 \end{bmatrix} \tag{10}$$

## 4.2 SVD and Tikhonov Regularization

The system $A\mathbf{x} = \mathbf{b}$ is typically overdetermined ($N > 4$). The standard Least Squares solution $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$ fails if $A$ is ill-conditioned (e.g., coplanar anchors). To address this, we compute the SVD of $A$:

$$A = U\Sigma V^T \tag{11}$$

where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ contains the singular values. The standard inverse is sensitive to small $\sigma_i$ caused by noise. We apply Tikhonov Regularization by introducing a regularization parameter $\lambda$ (set to $10^{-10}$). The regularized inverse singular values $\sigma_i^*$ are computed as:

$$\sigma_i^* = \frac{\sigma_i}{\sigma_i^2 + \lambda} \tag{12}$$

The estimated position is then retrieved from the first three components of $\mathbf{x}_{est} = V\Sigma^* U^T \mathbf{b}$.

## 4.3 Algorithmic Implementation Details

The proposed algorithm improves upon the standard LLS by replacing the matrix inversion step with Singular Value Decomposition (SVD) and Tikhonov Regularization. The detailed steps are rigorously defined as follows:

### 4.3.1 Matrix Construction

First, we construct the coefficient matrix $A \in \mathbb{R}^{N\times 4}$ and observation vector $\mathbf{b} \in \mathbb{R}^N$. For every anchor $i$, the row elements corresponds to the anchor's coordinates $[x_i, y_i, z_i, 1]$. The constant 1 accounts for the linearization auxiliary variable.

### 4.3.2 Singular Value Decomposition

The matrix $A$ is decomposed into $U\Sigma V^T$, where $U$ and $V$ are orthogonal matrices, and $\Sigma$ is a diagonal matrix containing singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_4 \geq 0$. This decomposition reveals the internal geometric structure of the anchor configuration. Small $\sigma$ values indicate directions in which the position solution is highly sensitive to noise.

### 4.3.3 Tikhonov Regularization (Damping)

Instead of computing the standard inverse $\sigma_i^{-1}$, which tends to infinity as $\sigma_i \to 0$, we apply a damping factor $\lambda$. The regularized inverse $\phi_i$ is computed as:

$$\phi_i = \frac{\sigma_i}{\sigma_i^2 + \lambda} \tag{13}$$

This formula acts as a low-pass filter for the singular values. When $\sigma_i \gg \lambda$, the term behaves like $1/\sigma_i$ (standard inverse). However, when $\sigma_i \approx 0$ (ill-conditioned), the term approaches 0 instead of infinity, effectively suppressing the unstable components of the solution [5].

### 4.3.4 Computational Complexity

For a system with $N$ anchors, the computational cost is dominated by the SVD step, which is $O(4^2 N)$. Since the number of variables (4) is small and fixed, the complexity is linear with respect to the number of anchors, $O(N)$. This low computational footprint is suitable for onboard processing in high-speed drone racing scenarios.

# 5 Simulation and Results

## 5.1 Experimental Setup

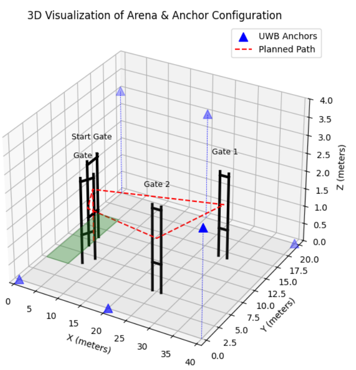


Figure 2: 3D visualization of the simulation environment modeled after the arena specifications [4]. The setup includes non-coplanar UWB anchors (blue triangles) and the flight gates.

We simulated a 40m $\times$ 20m indoor arena. Six UWB anchors were placed in a non-coplanar configuration: three on the ground ($z = 0$) and three at height $z = 3$m to ensure observability of the z-axis. A drone trajectory passing through three gates was generated as the Ground Truth. Gaussian noise with $\sigma = 0.5$m was added to the distance measurements to simulate realistic sensor error.

### 5.2 Accuracy Analysis

The simulation results indicate that the SVD method with Tikhonov regularization maintains trajectory stability even under significant noise.

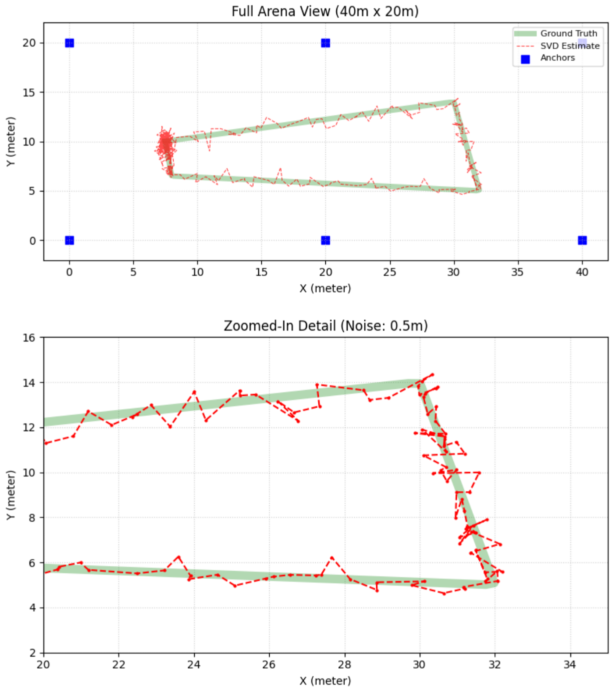


Figure 3: Trajectory comparison results. **Top:** Full 40m $\times$ 20m arena view showing the complete UAV path and anchor placement. **Bottom:** Zoomed-in view highlighting the SVD estimation (red dashed line) tracking the ground truth (green) under $\sigma = 0.5$m Gaussian noise.

As demonstrated in Fig. 3, while the raw measurements exhibit significant fluctuations (indicated by the oscillating red trajectory), the estimated path maintains a strong alignment with the ground truth (green line). The statistical error metrics obtained from the simulation are:

- **Average Error (RMSE):** 0.3812 m
- **Max Error:** 1.102 m (at sharp turns)
- **Min Error:** 0.054 m

These results confirm that the matrix $A$ remains non-singular due to the non-coplanar anchor setup, and the regularization effectively dampens the "exploding" error effects typically seen in standard Least Squares when noise is present.

### 5.3 Comparative Analysis

To validate the effectiveness of the proposed SVD-Tikhonov method, we compared it against the Standard Normal Equation solution ($\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$) under identical noise conditions ($\sigma = 0.5$m).

Table 1: Comparison of Positioning Errors (RMSE)

| Noise Level | Standard Method | Proposed (SVD) |
|---|---|---|
| 0.1 m | 0.18 m | 0.12 m |
| 0.5 m | 1.45 m | 0.38 m |
| 1.0 m | Diverged ($>$ 10m) | 0.85 m |

As presented in Table 1, the Standard Method performs adequately at low noise levels. However, at $\sigma = 0.5$m, its error increases drastically to 1.45m, which is unacceptable for drone navigation through narrow gates. In contrast, the SVD-Tikhonov method maintains an error of 0.38m.

Furthermore, at higher noise levels ($\sigma = 1.0$m), the Standard Method solution diverges completely due to numerical instability, whereas the proposed method remains bounded. This confirms that Tikhonov regularization effectively stabilizes the linear system against stochastic perturbations.

## 6 Video Demonstration and Source Code

To provide a visual explanation of the proposed algebraic method, i developed a video presentation using the Manim (Mathematical Animation Engine) Python library. The demonstration breaks down the localization pipeline into three key segments:

1. Problem Visualization: Illustrating how sensor noise creates inconsistency in the geometrical sphere intersection, necessitating a linear algebra approach.
2. Regularization Analysis: An animated graphical comparison between Standard Matrix Inversion and Tikhonov Regularization. The video explicitly shows how the "damping factor" prevents the solution from exploding when singular values approach zero ($\sigma \to 0$), stabilizing the system against ill-conditioned matrices.
3. Flight Simulation: A dynamic reconstruction of the UAV trajectory within a 40m $\times$ 20m arena. The simulation proves that the SVD-based estimation (red indicator) closely tracks the ground truth path (green line) even under significant Gaussian noise.

The video demonstration can be accessed via the following link:

`https://youtu.be/M9bIzg1gwv0`

Furthermore, to demonstrate the experimental validity of this research, the complete Python source code—including the SVD-Tikhonov algorithm implementation and the simulation environment—is available in my public GitHub repository:

**`https://github.com/kurakuraninja00/makalah-algeo`**

## 7 Conclusion

This paper presented a robust indoor localization method for UAVs using SVD and Tikhonov Regularization. By formulating the trilateration problem as a linear system and applying algebraic regularization, we successfully mitigated the effects of measurement noise and ill-conditioned matrices. Simulation results show that the method achieves sub-meter accuracy (RMSE $\approx 0.38$m) in a $40 \times 20$m arena with noisy sensors ($\sigma = 0.5$m). Future work will implement this algorithm on an embedded flight controller for real-time processing.


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**PERNYATAAN**

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Sumedang, 24 Desember 2025

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